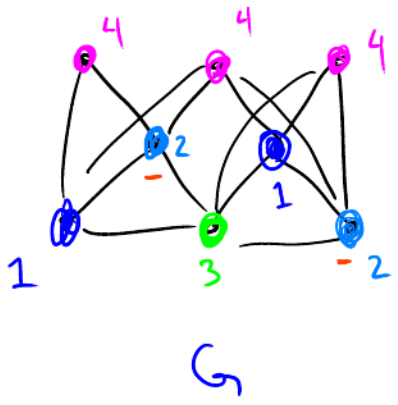
Last Time: Coloring Graph (properly)

Recall: $k : V(G) \to [k]$ is a (proper) coloring of graph G when for all $uv \in E(G)$ we have $k(u) \neq k(v)$.

$\chi(G)$ = minimum number of colors in a coloring.

Saw: Many examples...

Ex:



G

(NB: $K_4 \leq G$).

$4 \leq \chi(K_4) \leq \chi(G)$

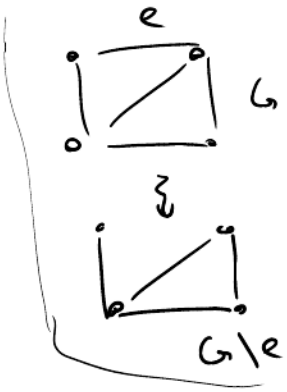
Moreover, k illustrated at the left is a proper coloring of G. ∎

Defn: Let G be a graph and $e \in E(G)$.

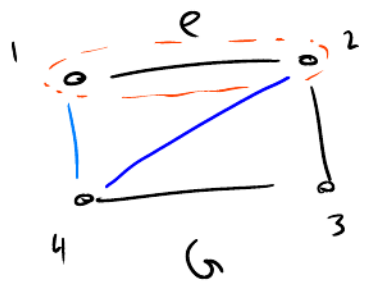
① The deletion of e from G is the graph $G \setminus e$ with $V(G \setminus e) = V(G)$ and $E(G \setminus e) = E(G) \setminus \{e\}$

② The contraction of e from G is the graph G/e w/ $V(G/e) = V(G)/_\sim$ and $E(G/e) = E(G) \setminus e$

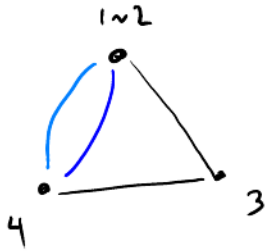
where $V(G)/_\sim$ is the set obtained by identifying the ends of e and edges are attached to the corresponding new vertices



Ex:



G



G/e

Q: What happens if we contract a loop (i.e. an edge connecting a vertex to itself)?

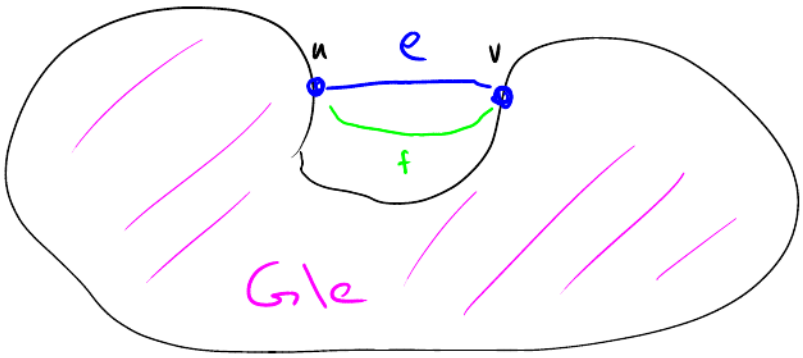
A: Same as deleting it!

Observation: G/e might not be a simple graph, even if G was a simple graph...

Ex:

Q: How do deletion and contraction relate to coloring?

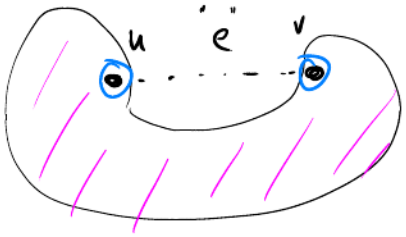
Given a graph G with edge e, let's color $G \setminus e$ ...



Case 0: if e is a loop, G had no proper coloring... ($k(u) = k(u)$).

Case 1: there is another edge f connecting u to v. Then every coloring of $G \setminus e$ is also a coloring of G (f forces $k(u) \neq k(v)$).

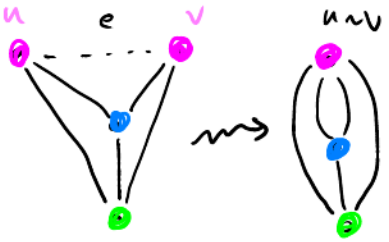
Case 2: If e is not a loop and there is no other edge connecting ends of e, then coloring $G \setminus e$ has two possibilities:

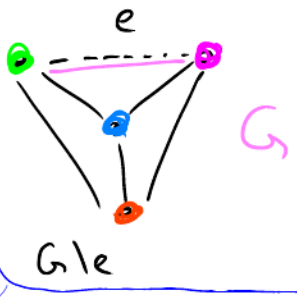


Given a coloring k of $G \setminus e$, we have either $k(u) = k(v)$ or $k(u) \neq k(v)$.

If $k(u) \neq k(v)$, then k is also a coloring of G.

If $k(u) = k(v)$, then k is also a coloring of G/e.





Point: Given a coloring of $G \setminus e$ where e is not a loop, we can use that coloring to build a new coloring of either G or G/e depending on the colors assigned to ends of e.

Prop: Let G be a graph (not necessarily simple). If $e \in E(G)$ is not a loop, then there is a bijection

$$\{c : c \text{ is a } k\text{-coloring of } G \setminus e\} \longleftrightarrow \{c : c \text{ is a } k\text{-coloring of } G\} \sqcup \{c : c \text{ is a } k\text{-coloring of } G/e\}.$$

Exercise: Formalize our argument from before.

Defn: The <u>chromatic function</u> of a graph G is the function

$$P_G : \mathbb{N}_0 \longrightarrow \mathbb{N}_0 \quad \text{defined by}$$

$$P_G(k) := \#\{c : c \text{ is a } k\text{-coloring of } G\}.$$

Ex: If $G = K_1$, $P_{K_1}(k) = k$.

k	0	1	2	3	...
$P_{K_1}(k)$	0	1	2	3	

pf: each function $f : \{v\} \to [k]$ is a coloring of K_1. There are k such functions (one for each element of $[k]$). ☑

$c : \{v\} \to \emptyset$ $\quad$ $c : \{v\} \to [1]$ $\quad$ $c : \{v\} \to [2]$

Ex: $G = K_2$ $\quad$ u • —e— • v .

We know every k-coloring of $G \backslash e$ $\left(\begin{smallmatrix} \bullet & \bullet \\ u & v \end{smallmatrix}\right)$ is either a coloring of G (if $c(u) \neq c(v)$) or it induces a coloring of G/e (if $c(u) = c(v)$).

$G \backslash e$

$P_{G \backslash e}(k) = k^2$

$P_G(k) = ?$

G

$P_{G/e}(k) = k$

G/e

pf: Coloring $G \backslash e$ amounts to making two independent choices: choose color of u, then the color of v. ☑

Hence $P_{G \backslash e}(k) = P_G(k) + P_{G/e}(k)$ (proposition earlier...), so $k^2 = P_G(k) + k$

$\therefore P_G(k) = k^2 - k$
$= k(k-1)$.

NB: We can determine $P_{K_n}(k)$ more directly using the product principle...

Prop: $P_{K_n}(k) = k(k-1)\cdots(k-(n-1))$ for all $k \in \mathbb{N}_0$ and all $n \geq 1$.

Sketch: Induct on n. $P_{K_{n-1}}(k) = k(k-1)\cdots(k-(n-2))$. Coloring K_n amounts to choosing a coloring of K_{n-1} and an additional color for the "new vertex". ☑

Let G be a graph. We can determine $P_G(k)$ via

$$P_G(k) = P_{G\setminus e}(k) - P_{G/e}(k)$$ for any edge e which is not a loop.

Prop: $P_G(k)$ is a polynomial function in k.

pf: We induct on the number of edges of G.

Base Cases: If $\#E(G) = 0$, then $P_G(k) = k^{\#V(G)}$.

If $\#E(G) = 1$, then $P_G(k) = \begin{cases} 0 & \text{if the edge is a loop} \\ k^{(\#V(G))-2} \cdot k(k-1) & \text{o/w.} \end{cases}$

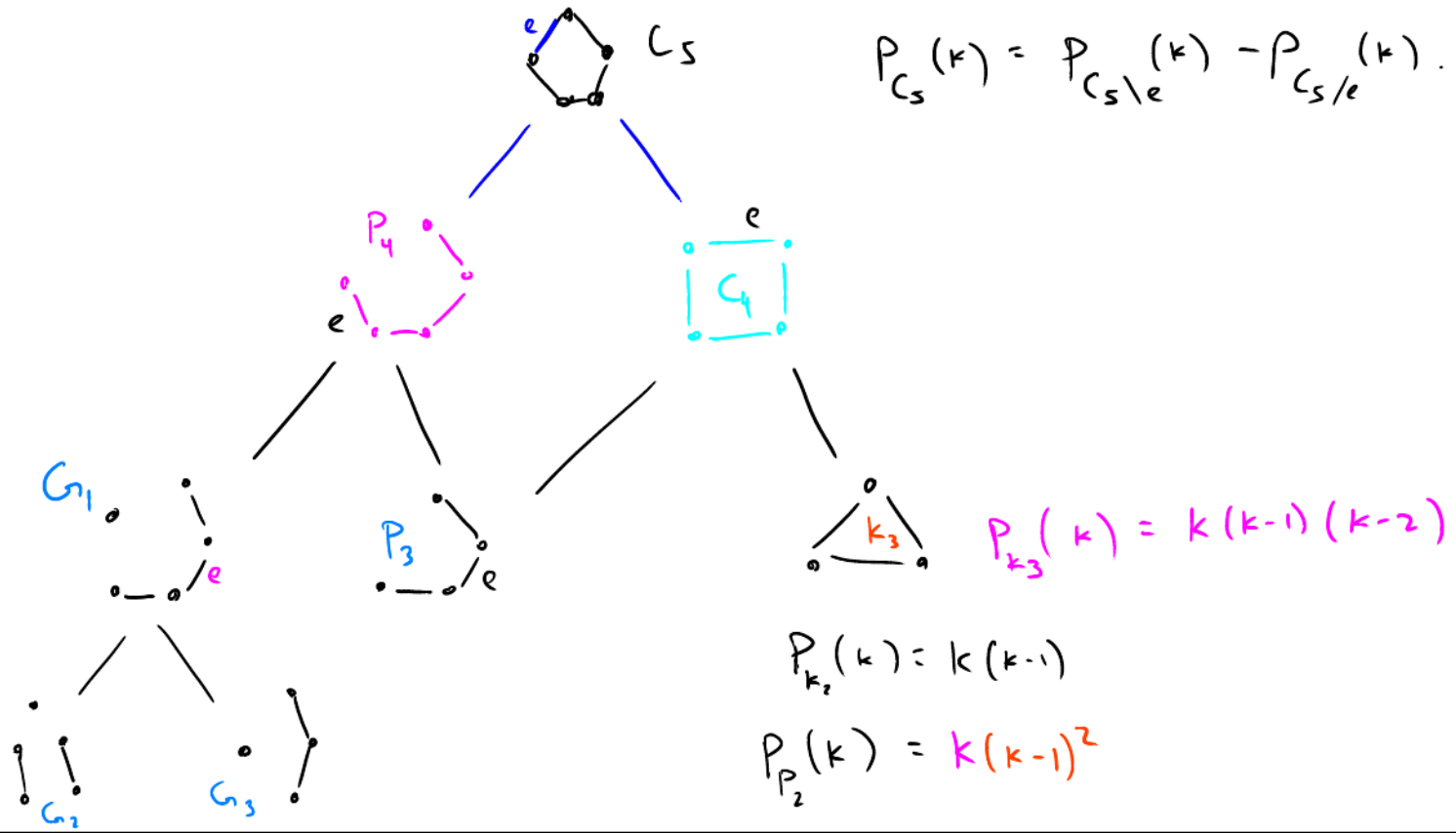
Both base cases are polynomials in k.

Inductive step: Supposing the result holds for all graphs with n edges, and supposing G has $n+1$ edges, we see

$$P_G(k) = P_{G\setminus e}(k) - P_{G/e}(k)$$ is a difference of polynomials, which is again a polynomial.

Hence the result holds. ∎

Ex: Compute $P_G(k)$ for $G = C_5$.



$$P_{C_5}(k) = P_{C_5\setminus e}(k) - P_{C_5/e}(k).$$

$$P_{K_3}(k) = k(k-1)(k-2)$$

$$P_{K_2}(k) = k(k-1)$$

$$P_{P_2}(k) = k(k-1)^2$$

Thus we compute $P_{C_5}(k)$ below (see above for labels):

$$P_{C_5}(k) = P_{P_4}(k) - P_{C_4}(k)$$

$$= \left(P_{G_1}(k) - P_{P_3}(k) \right) - \left(P_{P_3}(k) - P_{K_3}(k) \right)$$

$$= P_{G_1}(k) - 2P_3(k) + P_{K_3}(k)$$

$$= \left(P_{G_2}(k) - P_{G_3}(k) \right) - 2P_3(k) + P_{K_3}(k)$$

$$= \left(k \cdot k(k-1) \cdot k(k-1) - k \cdot k(k-1)^2 \right) - 2k(k-1)^2 + k(k-1)(k-2).$$

$$= k\left(k^2(k-1)^2 - k(k-1)^2 - 2(k-1)^2 + (k-1)(k-2) \right)$$

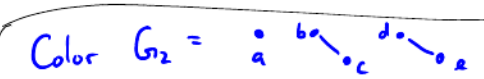
$$= k(k-1)\left(k^2(k-1) - k(k-1) - 2(k-1) + (k-2) \right)$$

$$= k(k-1)\left(k^2(k-1) - k(k-1) - 2k + 2 + k - 2 \right)$$

$$= k^2(k-1)\left(k(k-1) - (k-1) - 1 \right)$$

$$= k^2(k-1)\left(k(k-1) - k \right)$$

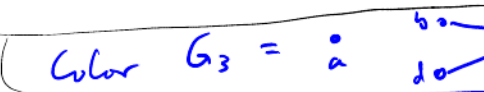
$$= k^3(k-1)(k-2)$$ ☺

Color G_2 = 

① Color a (k ways)

② Color b, c (k(k-1) ways)

③ Color d, e (k(k-1) ways)

Color G_3 = 

① Color a (k ways)

② Color b, c, d ($k(k-1)^2$ ways)